AN ANALYTIC SOLUTION OF SIMPLIFIED MATHEMATICAL MODEL OF EQUATION OF CONTINUITY (CONSERVATION OF FLOW EQUATION).

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**Abstract**

An analytic method of solving a partial differential equation (PDE) obtained from a modeled equation of continuity (conservation of flow equation) using integral transform method was proposed. The analysis of the equation was carried out with regards to the distance along road *(x)*, time *(t)* and traffic flow *(k)*. Whereas density of traffic *(k0)* and velocity *(u)* values were varied to asses how the vehicular flux changes in the study, one after the other keeping the rest fixed in order. The solution of the problem was discussed after analyzing the effect of the parameters on the traffic flow and graphs were presented to illustrate the exactness of the analytical solution where the behaviour of the traffic flux changes with distance for different initial densities and changes with distance for different initial velocities.

*Keywords*: Analytic solution, Continuum, Traffic flow, Integral transforms method.

1. **Introduction**

In the twentieth century, a new problem which has arisen is how road traffic is going to be organized so that the benefits of increased mobility can be fully enjoyed in human life and capital at the lowest price. The problem has many sides; constructional, educational, legal, administrative [10]. Heavy traffic flow becomes progressively important as our large cities population develop. Numerous researches of traffic flows have been carried out specifically over the past three decades, particularly with strong concern in the subject since the early 1990s, testifying to its currently observed importance [8].

From different points of view, traffic flow modeling process was previously studied and the same process was described in different mathematical methods used. The appearance we can see on our roads and streets become hard to choose an appropriate way of its mathematical derivation [9]. Different views was recognized in different authors work to the similar phenomena and are focusing on distinct aspects of the similar problem while they have an understanding on basic parameters of traffic flow like, the traffic flow average speed or traffic flow rate, traffic flow density, traffic capacity. Besides, for example in [15], a lot of separate studies into the use of models of traffic flow to deal with various engineering problems are carried out. A comparison of separate continuum models has drawn that some of scientific studies were based on the theory of fluid dynamic and gas, i.e. kinetic traffic flow theory. The ‘microscopic’ or ‘macroscopic’ traffic flow models can be studied using kinetic traffic flow theory [9].

The conservation laws were used in modeling of traffic flow by [10] and [13]. They claimed as follows: Considering a road with heavy traffic and one lane (One dimensional) so that a good approximation of continuum is described. Let the density of cars be denoted by . Then, conservation of the number of vehicles produces traffic hydrodynamics that is acceptable for continuum descriptions of some conserved quantity [7]. However, our traffic flow knowledge is insufficient and behind that of other modes of transport. Interest in modeling traffic has many sources. Examples include microscopic and macroscopic crowds related with transport systems [4, 17, 20], general spectator occasions and sporting [3], holy places [2, 15], political protests [18] and fire escapes [19]. Complete detail of situations cannot be presented here. Nevertheless, the above researches illustrate the various behavior connected with different situations. As shown by [11] and [12], the behavior of pedestrians or vehicles varies not only with their physical characteristics but also with their intention.

According to [8], there are two philosophies fundamentally distinct for crowd motion or traffic flow modeling. The first philosophy includes treating pedestrians or traffic flow as discrete entity and walks them generally in a computer simulation through the domain and the modeling philosophy grants flexibility. It is exceptionally well marched to be used for small traffic or crowds. The second philosophy is undoubtedly better in comprehending the rules governing the general flows behavior. It is valid only in crowds or traffic that is large and involves treating the crowd or traffic as a whole. Crowds or traffic are treated as (i) a fluid (presently rare), (b) a continuum that responds to local influences, or (c) as in this study, by assuming traffic as continuum move. Thus, a continuum model eases a major problem in comprehending crowds as noted by [5], and a crowd or traffic ought to be considered as an identity. The two philosophies have their place. They ought to be seen as supplementing each other. Moreover, Lighthill and Whitham (1955) theory is a continuum theory in which traffic density is considered rather than discrete vehicles [10].

1. **Purpose of The Study**

Fluid mechanics is a physical science concerned with the behaviour of fluid at rest and in motion. It combines two separate approaches; the empirical hydraulics by the hydraulicians and the classical hydrodynamics developed by mathematicians [1]. Hydraulics is mainly concerned with the motion of water. On the other hand, hydrodynamics is essentially a mathematical science dealing with flow analysis based on the concept of an ideal fluid, a fictitious fluid in which both viscosity and fluid compressibility are assumed absent or negligible [1].

The purpose of this study is to come up with a mathematical framework for understanding the mechanics of traffic flow in a simple domain. The primary purpose is to understand what principles underline the motion of traffic by a one-dimensional behavior found by Harold Greenberg for vehicle motion [6].

1. **Domain Description of Traffic Flow Mathematical Model**

The concept of mathematical modelling between speed, density and flow of vehicles in a simple domain by treating the traffic stream as a continuous fluid will be considered, where the simple domain is a one-lane (one dimensional flow) indicated below, (Fig. 1).



Figure 1: one dimensional flow

1. **Research Methodology**

The mathematical study of traffic flow and in particular vehicular traffic flow is done with the aim to get better understanding of these phenomena and to assist in prevention of traffic congestions.

In modelling traffic flow, gridlock, optimal time of traffic light, jam density (one lane full, one lane there is no cars), the disorders which includes the mechanical faults, state of minds of drivers and road topology are some problems considered.

This research will specifically consider the mathematical modelling of traffic flow in a simple domain. The domain is one lane (one dimensional flow) with no overtaking, the space (gap) between cars are equal, the velocity is constant and density *ρ*>>1.

In modelling vehicular traffic, the following assumptions are considered:

1. The entire variables are considered to be continuous functions.
2. Traffic moves with attributes of a fluid that is why is considered to be a continuum.
3. Traffic of vehicles can generally be treated as a continuum, provided the characteristic distance scale between vehicles is much less than the characteristic distance scale of the region in which the vehicle move. Also incompressibility condition (density remains constant, when it is subjected to high pressure and temperature), constant velocity and density *ρ*>>1 will be assumed.
4. The human sense (driver condition), road topology, vehicles being roadworthy not anticipating breakdown are part of the assumptions of the study.
5. In this work, traffic is assumed to behave like a continuous fluid. The method of fluid dynamics may then be used except for the lowest densities of traffic.
6. **Model Formulation**

5.1 Conservation of Flow Equation

An equation describing fluid flow is the equation of continuity (conservation of flow equation)

Where *q* is the traffic flow, vehicles per hours, as

Equation (1) becomes

Let the velocity be a function of density

Then,

Substituting (4) into (3) we have,

Where,

1. **Solution of the Model Equations (Solution** **Techniques**)

The solution techniques to be employed is integral transform method (Laplace transform) where partial differential equation (PDE) will be changed to ordinary differential equation (ODE). Moreover, the most general first order linear PDE (containing two independent variables) is of the form as in (8) was also used too in solution techniques.

Where *A(x,y),B(x,y),C(x,y)* and *R(x,y)* are given functions. Clearly, if either *A(x,y)* or *B(x,y)* is zero then the PDE may be solved straightforwardly as a first-order linear ODE, the only modification being that the arbitrary constant of integration becomes an *arbitrary function* of *x* and *y* respectively.

When the PDE contains partial derivatives with respect to both independent variables then, of course, we cannot employ the above procedure but must seek an alternative method the detailed procedure was fully explained in the solution of the equations of the same format in [14].

6.1 SOLUTION OF FLOW EQUATION BY INTEGRAL TRANSFORM METHOD

Equation (6) can be solved using integral transform method considering the conditions

Solving the equation

We’ve,

But is constant hence also

Hence,

For

So,

Hence,

1. **Asymptotic Behavior And Discussion Of The Results**

7.1 ANALYSIS OF THE CONTINUITY EQUATION (CONSERVATION OF FLOW EQUATION)

This section discusses the mathematical analysis of traffic flow together with the traffic velocity and some specific density of traffic in a simple domain.

The solution of conservation of flow equation was obtained. The earlier solution of the equation was determined base on the boundary conditions

For the continuity equation (conservation of flow equation) (6) the solution was obtained as

Though, we have much interest in parameters (velocity and density) which are going to be varied to see the response to changes in these operating conditions that will change the behaviour significantly.

7.2 ANALYSIS OF EQUATION (25)

For

The range for is obtained using by fixing the density and varying the initial velocity *u*, from *u*=*3* to *u=10.5* at an interval of 1.5, the flux behaviour is represented in fig 2. We have the corresponding respectively as

The result is

145

150

155

160

0.1

0.2

0.3

0.4

0.5

0.6

0.7

0.8

0.9

1

Traffic flux (k)

Distance along road (x)

Figure 2: The behaviour of the traffic flux as it changes with distance, for different initial velocities.

Whereas, by fixing the initial velocity and varying density from =145, to =195 at interval of 10 we have the response of the flux to these changes expressed in fig 3

The result is

145

150

155

160

0.1

0.2

0.3

0.4

0.5

0.6

0.7

0.8

0.9

1

Traffic flux (k)

Distance along road (x)

Figure 3: The behaviour of the traffic flux as it changes with distance, for different initial densities.

1. **SUMMARY OF THE RESULTS**

In the analysis of the study, we presented and analysed our problem concerning traffic flow with velocity and density in a simple domain (one dimensional fluid flow). The effects of changes of the operating parameters to the behaviour are determined and represented in Figures 2 and 3

Some values were used for computation of the results, where the values assigned previously are of two types; fixed and varied parameters. There is fixed range of values in obtaining the results for distance along road *(x)*, time *(t)* and traffic flow *(k)*. While velocity *(u)* and density of traffic *(k0)* values were varied one after the other keeping the rest fixed in order to asses how the vehicular flux changes in the study.

Various examples were considered as above, so as to get the best result that give us the best and effective traffic flow in the domain, depending on the value of parameters in the example or case considered. Two *(2)* different examples were considered, analyzed and presented in this work, for each of the Two *(2)* examples considered, there are some different results plotted using different values of parameters. The conservation of flow equation has two results from the two parameters (k0) and (u) it has.

Figure 2 shows the behaviour of the traffic flux as it changes with distance, for different initial velocities.

* The traffic flow in the domain responds to changes over the whole area of the distance along the road with different velocities.
* For a larger value of velocity (i.e. above the range), say u *= 10.5*, caused the traffic flow very fast.
* For a lower value of velocity (i.e. below the range), say *u =3*, caused the traffic flow low.
* The normal value of velocity u *=6* make the traffic flow low.
* The result shows that, traffic flow distribution in a normal domain will increase significantly from the lower to the higher value of velocity unless at u=6

Figure 3 shows the behaviour of the traffic flux as it changes with distance, for different initial densities.

* The traffic flow in the domain responds to changes over the whole area of the distance along the road with different densities.
* For a larger value of density (i.e. above the range), caused the traffic flow very fast.
* For a lower value of density (i.e. below the range), caused the traffic flow low.
* By comparing our normal value of density, with the one above and below the range, we found that the gap between the traffic flow is directly proportional.
* The result shows that, traffic flow distribution in a normal domain will increase significantly from the lower to the higher value of density.
* Their will be gridlock for very small density.

1. **CONCLUSION**

Based on the solutions of the model obtained in this work and results obtained from the application of the models using different values of parameters, the following conclusions were arrived as;

* From the results obtained, traffic velocity make the traffic flow to respond to changes easily (from the specified traffic velocities considered).
* From the results obtained, traffic density yields no relationship for values above or below the range in the two equatios, only one, the last has significant respond for the values.

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