**Results on generalized fuzzy soft topological spaces**

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**Abstract.**  In this manuscript, the concept of a generalized fuzzy soft point is introduced and some of its basic properties are studied. Also, the concepts of a generalized fuzzy soft base (subbase) and a generalized fuzzy soft subspace are introduced and some important theorems are established. Finally, we study the relationship between fuzzy soft set, intuitionistic fuzzy soft set, generalized fuzzy soft set and generalized intuitionistic fuzzy soft set are investigated.

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**1. Introduction**

 **M**ost of our real life problems in engineering, social and medical science, economics, environment etc. involve imprecise data and their solutions involve the use of mathematical principles based on uncertainty and imprecision. To handle such uncertainties, Zadeh [22] in (1965) introduced the concept of fuzzy set (FS) and fuzzy set operations. The analytical part of fuzzy set theory was practically started with the paper of Chang [6] in (1968) who introduced the concept of fuzzy topological spaces. However, this theory is associated with an inherent limitation, which is the inadequacy of the parametrization tool associated with this theory as it was mentioned by Molodtsov in [16]. In 1999, Molodtsov [16 ] introduced the concept of the soft set (SS) theory which is free from the above problems and started to develop the basics of the corresponding theory as a new approach for modeling uncertainties. Shabir and Naz [19] in (2011) studied the topological structures of soft sets. Intuitionistic fuzzy set theory was introduced by K. Atanassov 1986 [3]. In recent times, the process of fuzzification of soft set theory is rapidly progressed. In 2001, Maji et al [12] and Maji et al [13] combined the theory of SS with the fuzzy and intuitionistic fuzzy set theory and called as fuzzy soft set (FSS) and intuitionistic fuzzy soft set (IFSS). Topological structure of fuzzy soft sets was started (2011) by Tanay and Burc kandemir [20]. The study was pursued by some others [5,8,18]. In 2010, Majumdar and Samanta [14] introduced generalized fuzzy soft set (GFSS) and successfully applied their notion in a decision making problem. Yang [21] pointed out that some results of Majumdar and Samanta [14] are not valid in general. Chakraborty and Mukherjee [4] introduced generalized fuzzy soft union, generalized fuzzy soft intersection and several other properties of generalized fuzzy soft sets. Also, they introduced generalized fuzzy soft topological spaces, generalized fuzzy soft closure, generalized fuzzy soft interior and studied some of their properties. Arora and Garg [1,2] solved the MCDM problem and established the IFSWA operator and the IFSWG operator under the IFSS environment. Garg [9] introduced some series of averaging aggregation operators have been presented under the intuitionistic fuzzy environment. Garg and Arora [10,11] introduced distance and similarity measures for dual hesitant fuzzy soft sets in multi-criteria decision making problem, also they presented some generalized and Group-based Generalized intuitionistic fuzzy soft sets in decision-making.

 This paper is organized as follows : Section 2 describes the overview of FS , SS, FSS, GFSS, IFS IFSS and GIFSS theories. Section 3 defines the generalized fuzzy soft point. Section 4 presented the generalized fuzzy soft base and a generalized fuzzy soft subbase. Section 5 define a generalized fuzzy soft subspace, and Section 6 studies the relation between FS, SS, FSS, IFSS, GFSS and GIFSS. Finally, a concrete conclusion has been summarized insection7.

**2. Preliminaries**

 In this section, we present the basic definitions and results which will be needed in the sequel.

**Definition** **2.1** ([22])**.** Let be a non-empty set. A fuzzy set *A* in *X* is defined by a membership function : *X* [0,1] whose value represents the "grade of membership" of  *x* in *A* for *x X*. The set of all fuzzy sets in a set *X* is denoted by , where is the closed unit interval [0,1].

**Theorem** **2.2** ([22])**.** If *A, B* , then, we have:

(i)

(ii)

(iii)

(iv)

(v)

**Definition** **2.3** ([16])**.** Let be an initial universe and be a set of parameters. Let denotes the power set of *X* and . A pair *()* is called a soft set over *X* if *f* is a mapping from *A* into *P(X),* i.e,

In other words, a soft set is a parameterized family of subsets of the set *X*. For , may be considered as the set of approximate elements of the soft set

**Definition** **2.4** ([12])**.** Let *X* be an initial universe and *E* be a set of parameters. Let be the set of all fuzzy sets in *X* and . A pair is called a fuzzy soft set over *X*, where is a function, i.e, for each , is a fuzzy set in *X*.

**Definition** **2.5** ([14])**.** Let *X* be a universal set of elements and *E* be a universal set of parameters for *X*. Let and be a fuzzy subset of *E*, i.e, . Let be the mapping defined as follows:

, where and . Then is called a generalised fuzzy soft set (GFSS in short) over

**Definition** **2.6** ([14])**.** Let and be two GFSSs over . Now is said to be a GFS subset of or is said to be a GFS super set of if

(i) is a fuzzy subset of ;

(ii) is also a fuzzy subset of .

In this case we write .

**Definition** **2.7** ([14])**.** Let be a GFSS over The complement of , denoted by , is defined by , where and*,* .

Obviously ( .

**Definition** **2.8** ([4])**.** Let and be two GFSSs over . The union of and , denoted by , is The GFSS , defined as such that

 , where and .

 Let (, where is an index set, be a family of GFSSs. The union of these family , denoted by , is The GFSS , defined as such that , where (, and(, .

**Definition** **2.9** ([4])**.** Let and be two GFSSs over *.* The Intersection of and , denoted by , is The GFSS , defined as such that

, where and , .

 Let (, where is an index set, be a family of GFSSs. The Intersection of these family, denoted by , is the GFSS , defined as such that , where (, and (, .

**Definition** **2.10** ([14])**.** A GFSS is said to be a generalized null fuzzy soft set, denoted by , if such that where and ( Where ).

**Definition** **2.11** ([14])**.** A GFSS is said to be a generalized absolute fuzzy soft set, denoted by , if , where (e)is defined by and ( Where ).

**Definition** **2.12** ([4])**.** Let *T* be a collection of generalized fuzzy soft sets over *.* Then *T* is said to be a generalized fuzzy soft topology (GFST, in short) over if the following conditions are satisfied:

(i) and are in *T*;

(ii) Arbitrary unions of members of *T* belong to *T*;

(iii) Finite intersections of members of *T* belong to *T*.

The triplet is called a generalized fuzzy soft topological space (GFST- space, in short) over *.*

The members of *T* are called a GFS open sets in . The complement of a GFS open set is called GFS closed.

**Definition** **2.13** ([4])**.** Let be a GFST-space and be a GFSS over Then the generalized fuzzy soft closure of , denoted by , is the intersection of all GFS closed supper sets of

 Clearly, is the smallest GFS closed set over which contains .

**Definition** **2.14** ([4])**.** Let be a GFSS over . We say that read as belongs to the GFSS if and , .

**Definition** **2.15** ([4])**.** A GFSS in a GFST-space is called a generalized fuzzy soft neighborhood [GFS-nbd, in short] of the GFSS if there exists a GFS open set such that .

**Definition** **2.16** ([4])**.** A GFSS in a GFST-space is called a generalized fuzzy soft neighborhood of the generalized fuzzy soft point if there exists a GFS open set such that *.*

**Definition** **2.17** ([17])**.** Difference of two GFSS and , denoted by , is a GFSS , defined as and, .

**Definition 2.18 ([3]).** Let a set be fixed. An ***intuitionistic fuzzy set or***  in is anobject having the form where the functions & define the degree of membership and non-membership, respectively, of the element to the set & for every , .

**Definition 2.19 ([13]).** Let be an initial universe and be a set of parameters. Let be the set of all intuitionistic fuzzy subsets of and Then, the pair is called an intuitionistic fuzzy soft set over , where is a mapping given by .

For any , is an intuitionistic fuzzy subset of . Let us denote and by the membership degree and non-membership degree, respectively, that object holds parameter , where and . Then, can be written as an intuitionistic fuzzy set such that .

**Definition** **2.20** ([7])**.** Let be an initial universe and be a set of parameters. Let be the set of all intuitionistic fuzzy subsets of and Let be a mapping given by and be a mapping given by Let be a mapping given by and defined by

 ,

where and . Then, the pair is called a ***generalized intuitionistic fuzzy soft set*** over .

**3. Generalized fuzzy soft points and neighborhood systems**

In this section we introduce a generalized fuzzy soft point and study some of its basic properties Also, we discuss the concept of a neighborhood of a generalized fuzzy soft point in a generalized fuzzy soft topological space.

**Definition** **3.1.** The generalized fuzzy soft set GFS is called generalized fuzzy soft point ( GFS point in short) if there exists the element and such that and for all and . We denote this generalized fuzzy soft point .

 and are called respectively, the support and the value of .

**Definition** **3.2.** The complement of a generalized fuzzy soft point , denoted by , is defined as follows .

**Example 3.3.** Let and the set of parameters. Then

{} is generalized fuzzy soft point whose complement is **{** **}**.

 We redefine the belongingness of a generalized fuzzy soft point to a generalized fuzzy soft set in Definition (2.14) as follows :

**Definition** **3.4.** Let be a GFSS over *.* We say that read as belongs to the GFSS if for the element, and

**Definition** **3.5.** Let and be two generalized fuzzy soft points, we say that and *,* .

**Theorem** **3.6.** Let be a GFSS over *,* then:

 (1) ;

 (2) ;

 (3) if is not hold;

 (4) .

**Example** **3.7.** Let and , consider the GFSS over , as :

,

. Then

 ,

 .

Consider the GFS point , then

 and . For Theorem 3.6 (4). We have

 but .

**Definition** **3.8.** Let be a GFST-space. The set of all GFS neighborhoods of a generalized fuzzy soft point is called the GFS neighborhoods system of and is denoted by .

**Theorem 3.9.** Let GFSS (*X,E*) be a family of all generalized fuzzy soft sets over soft universe and be GFST-space. Then the following properties are satisfied:

(1) ;

(2);

(3) ;

(4) and .Then ;

(5) ⟹ such that for each .

**Proof**

(1) straightforward.

Let , Then there exists such that .Therefore, *,*  put and ,

then and . This shows that .

(2) Let .

Then and . Then there exists such that and which shows that .

Conversely, let such that for some Thus and which implies that

 and .

Therefore and . So .

(3) Let , then . Therefore

 and . Thus

 , and . Therefore .

 Conversely Let , then , and . Therefore and .This implies that

 and . Hence .

(4) Let ,Then .This implies that

 and , which implies that

 and . Therefore

 and . Then .

(5) Let , then there exists a generalized fuzzy soft set such that . Put . Then for every , . This implies that .

**Example 3.10.**

 We give example GFS neighborhood of GFS set and GFS point which are defined (2.15, 2.16).

Let . Consider the following GFSSs over defined as

, . Consider , . Then *T* forms GFS topology over *.*

Consider the following GFSS over ,

If then there exists GFS open set such that , i .e, is a GFS neighborhood of . Also, if , then is a GFS neighborhood of , where GFSS the family of all generalized fuzzy soft sets over .

**(4). Generalized fuzzy soft base and generalized fuzzy soft subbase**

**Definition 4.1.** Let be GFS topological space**.** A collection of generalized fuzzy soft sets over is called a generalized fuzzy soft open base or simply a base for generalized fuzzy soft topology on , if the following conditions hold:

1

2i.e. for each and , there exists such that

 and .

3 If , then for each and , there exists such that

 and and .

**Example 4.2.** Let . Let us consider the collection . Where

 { },

{ },

{ },

{ },

{ },

{ }.

We can see that satisfies the conditions (1-3) of Definition (4-1).Therefore forms a GFS base for a topology on .

**Definition 4.3.** Let be a GFS base for a GFS topology on . Then the GFS topology generated by GFS base , is denoted by and is defined as follows

 .

**Example 4.4.** Let  , and , where

,

,

 .

Then obviously, is a GFS base for a GFS topology on . The GFS topology generated by is , where

 ,

{, ,

,

**}**.

**Theorem 4.5.** Let *(* be a GFST-space and be a sub collection of such that every member of is union of some members of . Then is GFS base for the GFS topology on.

**Proof**

Since , . Again since , Let (. Then ( and so (. Then there exists , such that

 ( *:* }. Therefore

( *:* }, for . That is, for each

 min { (}max{( : } and

 min { (}max{ ( : }.

Therefore there exists such that

 min { (} ( and min{ (} (.

Thus for , we get such that ( and

 min { } and min{ (} (.

Therefore is GFS base for the GFS topology on

**Definition** **4.6.** A collection of some members of GFST-space ( is said to be a subbase of if and only if the collection of all finite intersection of members of is a base for.

**Example 4.7.** Let and , where

 { },

{ },

{

The collection of all finite intersection of members of is the base in Example (4.2). So is a subbase for a GFS topology on.

**Theorem** **4.8.** A collection of GFSSs over is a subbase for a suitable GFS topology if and only if

 (1) or is the intersection of finite number of members of .

 (2) .

**Proof**

First let be a subbase for and be a base generated by . Since , either or is expressible as an intersection of many finite members of . Now let and. Since , there exists such that .

Since there exists Such that

 *.*

Therefore , and so

 for some , , for some *.*

Thus and . Hence .

 Conversely, let be collection of GFSSs over satisfying the condition (1) and (2). Let be the collection of all finite intersection of members of . Now it enough to show that forms base for suitable GFS topology. Since is the collection of all finite intersection of members of , by assumption (1) we get and by (2) we get

 . Again let and . Since , there exists , for such that . Again since , there exists ,

for such that . Therefore

 .

That is, . This completes the proof.

 **5. Generalized fuzzy soft topological subspaces**

**Definition 5.1.** Let be a GFS topological space . Let be an ordinary subset of and be GFSS over such that

 , ,

i.e. ,.,

Let }. We can show that is a GFS topology on as follows :

(i) Since , , and , then , .

(ii) Suppose that , . Then for each , there exist, such that . We have

 .

Since , we have

(iii) Let be a subfamily of . Then for each , there is a GFSS

 of such that . We have

 .

 Since , we have .

 is called the GFS subspace topology on and is called a GFS subspace of .

**Example** **5.2.** Let, . Consider and as follows:

,

*.*

We consider the GFS topology on as ,

Let .

 , where

(i)

(ii)

(iii) ,

(iv) { ,

Thus The collection is GFS a topology on *.*

**Theorem**  **5.3.** Let be a GFS subspace of and a GFSS over . Then

(i) is GFS closed in if and only if for some GFS closed set in .

(ii) where is the closure of in with respective to .

**Proof**

(i) If is GFS closed in then we have

 , for some .

 Now, , for some .

 . (since , see Definition 5.1)

 where is GFS closed in as .

 Conversely, assume that for some GFS closed set in . This

mains that .

 Now, and hence is GFS closed in .

(ii) We have, is a GFS closed set in . Then is a GFS closed set in . Now  and GFS closure of in is the smallest GFS closed set containing , so.

On other hand where is GFS closed in *.*Then is GFS closed set containing and so . Therefore .

It is useful to investigate the relationship between types of sets that are generalized fuzzy set.

**6. Generalized fuzzy soft set(**GFSS)**& Intuitionistic fuzzy soft set(**IFSS**)**

**Lemma 6.1.** The relationships between the sets: FS, SS, FSS, IFSS, GFSS and GIFSS that are generalized the crisp set (CS) notion are illustrated in the following diagram

CS

FS

SS

IFS

FSS

GFSS

IFSS

GIFSSSS

**The converse of the arrows of the above diagram need not be true.**

**Counter example 6.2.** Let , and .

(1) Let

, is an IFSS but not GFSS.

(2) Let ,

,

 is GFSS but not IFSS.

 (3) Let

, is GIFSS but neither IFSS or GFSS.

 From the above example , we see that GFSS and IFSS are independent notions

**Lemma 6.3.**Similarly one can be deduce similar Diagram previous the relationship between analogues topologies.

**7. Conclusion**

 In this paper, we have introduced generalized fuzzy soft point, generalized fuzzy soft open base and subbase. The generalized fuzzy soft topological subspaces is introduced. Finally, we cocluded that GFSS and IFSS are independent notions , whereas each of them is GIFSS. So, one can try to introduce some special properties of compactness, some separation axioms, connectednessetc, on generalized fuzzy soft topological spaces.

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