Partial Skew generalized Power Series Rings

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Abstract

In this paper, using generalized partial skew versions of Armendariz rings, we study the transfer of left (right) zip property between a ring $R$ and partial skew generalized power series rings over $R$.

1 Introduction

A group $G$ acts on a set $X$ if for every $g \in G$ there exist a bijection $\alpha_g$ of $X$ such that $\alpha_1 = id_X$ and $\alpha_g \circ \alpha_h = \alpha_{gh}$, for any $g, h \in G$. where 1 denotes the identity element of $G$. If $X = R$ is a ring, then $G$ acts on $R$ if moreover, any $\alpha_g$ is an automorphism of $R$, for all $g \in G$.

Partial action of groups appeared independently in various areas of mathematics, in particular, in the theory of operator algebras giving a powerful tool of their study (see [10], [12], [13] and [14]).

Recently, in a pure algebraic context, partial representations and partial actions of groups on algebras have been introduced and studied in [10] and [11]. Also a Galois theory of commutative rings with partial actions have been developed in [12].

Definition 1 [9] Let $G$ be a group and $R$ a unital $K$- algebra, where $K$ is a commutative ring. a partial action $\alpha$ of $G$ on $R$ is a collection of ideals $D_g$, $g \in G$ of $R$ and isomorphisms of (non-necessarily unital) $K$-algebras $\alpha_g : D_g^{-1} \to D_g$ such that :

(i) $D_1 = R$ and $\alpha_1$ is the identity mapping of $R$;

(ii) $D_{(gh)^{-1}} \supseteq \alpha_h^{-1}(D_h \cap D_{g^{-1}})$;

(iii) $\alpha_g \circ \alpha_h(a) = \alpha_{gh}(a)$, for any $a \in \alpha_h^{-1}(D_h \cap D_{g^{-1}})$
Note that conditions (ii) and (iii) mean that the function $\alpha_{gh}$ is an extension of the function $\alpha_g \circ \alpha_h$. The property (ii) easily implies that $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$, for all $g, h \in G$. Also $\alpha_{g^{-1}} = \alpha_g^{-1}$ for every $g \in G$.

Natural examples of partial actions can be obtained by restricting a global action to an ideal. More precisely, suppose that a group $G$ acts on an algebra $T$ by automorphisms $\sigma_g : T \to T$ and let $I$ be an ideal of $T$. Assume that $D_g = I \cap \sigma_g(I) \neq 0$ for every $g \in G$. Define $\alpha_g$ as the restriction of $\sigma_g$ to $D_{g^{-1}}$. Then it is easy to verify that $\alpha = \{ \alpha_g : D_{g^{-1}} \to D_g \mid g \in G \}$ is a partial action of $G$ on $I$. We say, in this case, that $\alpha$ is the restriction of $\sigma$ to $I$. Thus it is natural to ask when a given partial action can be obtained as the restriction of a global action.

In [10, Definition 4.2] M. Dokuchaev and R. Exel showed that if a partial action $\alpha$ of a group $G$ on $T$ is given, then an enveloping action of $\alpha$ was defined as an extension algebra $R$ of $T$ such that $T$ is an ideal of $R$ together with a global action $\sigma = \{ \sigma_g \mid g \in G \}$ of $G$ on $R$, where $\sigma_g$ is an automorphism of $R$, such that $\alpha$ is the restriction of $\sigma$ to $T$ and the following properties hold:

1. $R = \sum_{g \in G} \sigma_g(T)$.
2. $D_g = T \cap \sigma_g(T)$, for every $g \in G$.
3. $\alpha_g(a) = \sigma_g(a)$, for all $g \in G$ and $a \in D_{g^{-1}}$.

They also showed that a partial action $\alpha$ has an enveloping action if and only if all ideals $D_g$ is generated by a central idempotent of $T$, for any $g \in G$ [10, Theorem 4.5].

Throughout this article, $R$ will denote an associative $K$-algebra with identity element $1_R$, $(G, +, \leq)$ is a strictly ordered additive group and $\alpha = \{ \alpha_g : D_{-g} \to D_g \}$ is a partial action of $G$ on $R$. We assume, unless other wise stated, that the partial action has an enveloping action denoted by $(S, \sigma)$. Then any of the ideals $D_g$ is generated by a central idempotent of $R$ which we denote by $1_y$. By the above condition 2, we have that $1_y = 1_R \sigma_g(1_R)$, where $\sigma_g(1_R)$ are central element in $S$. This fact and conditions (1) and (3) will be used freely in this article. Also a $K$-algebra will be called frequently simply a ring, and the following remark will be used without further mention: if $I$ is an ideal of $R$, then $I$ is also an ideal of $S$ (if $a \in I$ and $s \in S$ we have $sa = s1_Ra \in Ra \subseteq I$, and similarly $as \in I$). Note that $S$ does not necessarily have an identity, since the group acting on $R$ is infinite. A left (right) annihilator of a subset $U$ of $R$ is defined by $l_R(U) = \{ a \in R : aU = 0 \}$ ($r_R(U) =$
{a ∈ R : Ua = 0}). For a ring R put \( l_R(2^R) = \{l_R(U) : U ⊆ R\} \) and \( r_R(2^R) = \{r_R(U) : U ⊆ R\} \).

**Definition 2** The partial skew generalized power series is the set \( \Lambda \) of all maps \( \mu : G → R \), where \( \mu(g) \in D_g \) such that \( \text{supp}(\mu) = \{g ∈ G : \mu(g) ≠ 0\} \) is Artinian and narrow subset of \( G \), i.e. every strictly decreasing sequence of elements of \( \text{supp}(\mu) \) is finite and every subset of pairwise order incomparable elements of \( \text{supp}(\mu) \) is finite, with pointwise addition and product operation called convolution defined by \( (\muν)(g) = \sum_{(s,t) ∈ X_\mu(\nu)} α_s(\alpha_s(\mu(s))ν(t)) \) for each \( μ, ν ∈ \Lambda \), where \( X_\mu(\nu) = \{(s,t) ∈ G × G : s + t = g, \ μ(s) \text{ and } ν(t) ≠ 0\} \).

Since \( \text{supp}(\muν) ⊆ \text{supp}(\mu) + \text{supp}(ν) \). Then by [29, 2.1] \( \text{supp}(\muν) \) is Artinian and narrow which implies that \( \muν ∈ \Lambda \). Hence, \( \Lambda = [[R^G, α]] \) becomes a ring called the partial skew generalized power series with coefficient in \( R \) and exponent in \( G \), with the identity map \( ε : G → R \) defined by \( ε(0) = 1 \) and \( ε(g) = 0 \) for each nonzero \( g ∈ G \). Note that the partial skew generalized power series ring is not an associative algebra in general, but it is an associative algebra in case the partial action \( α \) has an enveloping action.

The mappings \( ε_\mu ∈ \Lambda \), defined by \( ε_\mu(g) = 1_g \) and \( ε_\mu(h) = 0 \) for each \( h ≠ g \) is an embedding of \( G \) into the multiplicative monoid of \( \Lambda \). Also, \( R \) canonically embedded as a subring of \( \Lambda \) via \( r → c_r \) such that \( c_r(0) = r \) and \( c_r(g) = 0 \) for each nonzero \( g ∈ G \). So, \( R ≅ c_R \) and we can identify \( r ∈ R \) with \( c_r ∈ \Lambda \).

Let \( π(μ) \) denote the set of all minimal elements of \( \text{supp}(μ) \). Clearly, if \((G, ≤)\) is totally ordered, then \( π(μ) \) contains of only one element which is still denoted by \( π(μ) \).

So, the structure of partial skew generalized power series ring generalize many classical constructions such as (partial skew and skew) polynomial ring, (partial skew and skew) Laurent polynomial ring, (partial skew and skew) power series ring, (partial skew and skew) group ring and formal power series ring.

Let \( T_\mu = C(μ) \) be the content of \( μ \) i.e. \( C(μ) = \{μ(g) : g ∈ \text{supp}(μ)\} \). Since \( R ≅ c_R \) we can identify, the content of \( μ \) with \( c_{C(μ)} = \{c_{μ(s)} : s ∈ \text{supp}(μ)\} ⊆ Λ \).

**2 Partial skew generalized power series over zip ring**

A ring \( R \) is called Armendariz if whenever the polynomials \( f(x) = \sum_{i=0}^{n} a_i x^i \) and \( g(x) = \sum_{j=0}^{m} b_j x^j \) in \( R[x] \), satisfy \( f(x)g(x) = 0 \) implies
that $a_i b_j = 0$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$. Such rings were introduced by Armendariz in [3] showing that reduced rings satisfy the above condition. Properties and algebraic structure of Armendariz rings have been studied extensively by many authors see for example [1], [3], [23] and [26].

Faith in [16] called a ring $R$ right zip if the right annihilator $r_R(X)$ of a subset $X$ of $R$ is zero, then $r_R(Y) = 0$ for a finite subset $Y \subseteq X$; equivalently, for a left ideal $L$ of $R$ with $r_R(L) = 0$, there exists a finitely generated left ideal $L_1 \subseteq L$ such that $r_R(L_1) = 0$. A ring $R$ is called zip if it is right and left zip. He also proved in [15] that if a ring $R$ is a commutative ring and $G$ is a finite abelian group, then the group ring $RG$ of $G$ over $R$ is a zip ring.

The concept of zip rings was initiated by Zelmanowitz [27] and appeared in various papers, see for example [5], [6], [15], [16].

Extensions of zip rings were studied by several authors. Beachy and Blair [5] studied rings which satisfy the condition that every faithful right ideal $I$ of $R$ is cofaithful in the sense that $r_R(I_1) = 0$ for a finite subset $I_1 \subseteq I$. In addition, they showed that if $R$ is a commutative zip ring, then the polynomial ring $R[x]$ over $R$ is zip. Hong et. al. [21] proved that if $R$ is an Armendariz ring, then $R$ is a right zip ring if and only if $R[x]$ is a right zip ring. They showed that a ring $R$ is right zip if and only if $Aut(M_n(R))$ is a right zip ring. Also, they studied monoid rings over zip rings and proved that $R$ is a right zip ring if and only if the monoid ring $RG$ is a right zip ring, when $R$ is reduced and $G$ is a u. p. monoid. In [8], W. Cortes generalized the above results to skew polynomial extensions over zip rings by using skew versions of Armendariz rings. In other words, he has shown that if $\sigma$ is an automorphism of $R$ and $R$ satisfies $SA1'(SA2')$ condition, then $R$ is a right zip ring if and only if $R[x, \sigma] (R[[x, \sigma]])$ is a right zip ring iff $R[x, x^{-1}, \sigma] (R[[x, x^{-1}, \sigma]])$ is a right zip ring. M. Baser [4] proved, under the assumption that $R$ is an $\alpha$ rigid ring, that a ring $R$ is a right zip ring if and only if the Ore extension $R[x, \alpha, \delta]$ is a right zip ring, where $\alpha$ is an endomorphism and $\delta$ is an $\alpha-$derivation of a ring $R$. In [6] Cedo studied matrices over zip rings and proved that if $R$ is a commutative zip ring, then $M_n(R)$ is a zip ring.

In this article, we study partial skew generalized power series over zip rings, using the ideas of [8] and [28].

**Definition 3** [28] Let $R$ be a ring, $G$ be a group and $\alpha$ be a partial action of $G$ on $R$. We say that $R$ is partial $\alpha$-compatible if for each $a, b \in R$ we have $ab = 0$ if and only if $a\alpha_s(b_1) = 0$ for each $s \in G$.

**Definition 4** If $R$ is a ring and $G$ is a strictly ordered group, then the ring $R$ is called a partial skew generalized Armendariz ring if for any
Lemma 5 Suppose that $R$ is an $\alpha$-compatible ring, $G$ is a strictly ordered group, $\alpha$ is a partial action of $G$ on $R$ and $\Lambda = [[R^{G,\alpha}]]$ is a partial skew power series ring. If $U$ is a subset of $R$, then

1. $r_\Lambda(U) = r_R(U)\Lambda$.
2. $l_\Lambda(U) = \Lambda l_R(U)$.

Proof.

1. Let $\mu \in r_\Lambda(U)$, hence $0 = c_\mu\mu$ for each $u \in U$. So, $(c_\mu\mu)(s) = \alpha_e(\alpha_{-e}(u)(\mu(s))) = u\mu(s)$ for each $s \in \text{supp}(\mu)$. Consequently, $\mu(s) \in r_R(U)$ for each $s \in \text{supp}(\mu)$. Hence, $\mu \in r_R(U)\Lambda$ and it follows that $r_\Lambda(U) \subseteq r_R(U)\Lambda$.

Conversely, suppose that $\mu \in r_R(U)\Lambda$, which implies that $0 = U\mu(s)$ for each $s \in \text{supp}(\mu)$. So, for each $u \in U$, $0 = u\mu(s) = \alpha_e(\alpha_{-e}(u)(\mu(s))) = (c_\mu\mu)(s)$ Hence $\mu \in r_R(U)\Lambda$ and it follows that $r_R(U)\Lambda \subseteq r_\Lambda(U)$. Consequently, $r_\Lambda(U) = r_R(U)\Lambda$.

2. Let $\mu \in l_\Lambda(U)$. So for each $u \in U$, $\mu c_u = 0$ and for each $s \in \text{supp}(\mu)$, $0 = (\mu c_u)(s) = \alpha_s(\alpha_{-s}(\mu(s)))u = \alpha_{-s}(\mu(s))u$. Since $R$ is partial $\alpha$-compatible we get $\mu(s)u = 0$ for each $s \in \text{supp}(\mu)$. Hence $\mu(s) \in l_R(U)$ for each $u \in U$. So, $\mu \in \Lambda l_R(U)$ and $l_\Lambda(U) \subseteq \Lambda l_R(U)$.

Conversely, let $\mu \in \Lambda l_R(U)$, so for each $s \in \text{supp}(\mu)$ and $u \in U$, $0 = \mu u = \alpha_s(\alpha_{-s} (\mu(s)))u = (\mu c_u)(s)$. Hence $\mu \in l_\Lambda(U)$ and it follows that $\Lambda l_R(U) \subseteq l_\Lambda(U)$. Consequently $l_\Lambda(U) = \Lambda l_R(U)$.

The above lemma gives us the maps $\phi : r_R(2^R) \rightarrow r_\Lambda(2^\Lambda)$ defined by $\phi(I) = I\Lambda$ for every $I \in r_R(2^R)$ without any condition on $R$ and $\Psi : l_R(2^R) \rightarrow l_\Lambda(2^\Lambda)$ defined by $\Psi(I) = \Lambda I$ for every $I \in l_R(2^R)$. Obviously $\phi$ is injective.

In the following lemma we show that $\phi$ is a bijective map if and only if $R$ is a partial skew Armendariz ring.

Lemma 6 Let $R, G, \alpha$ and $\Lambda$ be as in Lemma 5. Then the following are equivalent:

1. $R$ is a partial skew generalized Armendariz ring.
2. $\phi : r_R(2^R) \rightarrow r_\Lambda(2^\Lambda)$ defined by $\phi(I) = I\Lambda$ is a bijective map.
3. \( \Psi : l_R(2^R) \rightarrow l_\Lambda(2^\Lambda) \) defined via \( \phi(J) = \Lambda J \) is a bijective map.

**Proof.** Let \( Y \subseteq \Lambda \) and \( T = \bigcup_{\mu \in Y} T_\mu = \bigcup_{\mu \in Y} C(\mu) = \bigcup_{\mu \in Y}\{\mu(s) : s \in \text{supp}(\mu)\} \)

1. \( \rightarrow 2. \): Using Lemma 5 it is sufficient to show that \( r_\Lambda(\mu) = r_R(C(\mu))\Lambda \) for each \( \mu \in Y \). So, let \( \nu \in r_\Lambda(\mu) \), it follows that \( \mu \nu = 0 \). Since \( R \) is a partial skew generalized Armendarize ring, then for each \( x = s + t \) such that \( s \in \text{supp}(\mu) \) and \( t \in \text{supp}(\nu) \), \( 0 = \alpha_s(\alpha_{-s}(\mu(s))\nu(t)) \), hence \( \mu(s)\nu(t) = 0 \). So for a fixed \( s \in \text{supp}(\mu) \) and each \( t \in \text{supp}(\nu) \), we get \( \mu(s)\nu(t) = 0 \), and it follows that \( \nu \in r_R(\mu(s))\Lambda \). Consequently, for each \( s \in \text{supp}(\mu) \), \( \nu \in r_R(C(\mu))\Lambda \). Therefore \( r_\Lambda(\mu) \subseteq r_R(C(\mu))\Lambda \).

Conversely, let \( \nu \in r_R(C(\mu))\Lambda \). Hence \( C(\mu)\nu = 0 \), that is \( \mu(s)\nu = 0 \) for each \( s \in \text{supp}(\mu) \), which implies \( \mu(s)\nu(t) = 0 \) for each \( s \in \text{supp}(\mu) \) and \( t \in \text{supp}(\nu) \). Since \( R \) is partial \( \alpha \)-compatible, then \( 0 = \sum_{(s,t) \in X_s(\mu,\nu)} \alpha_s(\alpha_{-s}(\mu(s))\nu(t)) \) for each \( s \in \text{supp}(\mu) \) and \( t \in \text{supp}(\nu) \). So, \( (\mu\nu)(s) = \sum_{(s,t) \in X_s(\mu,\nu)} \alpha_s(\alpha_{-s}(\mu(s))\nu(t)) = 0 \). Therefore, \( \nu \in r_\Lambda(\mu) \) and it follows that \( r_R(\mu) \Lambda \subseteq \bigcap_{\mu \in Y} r_\Lambda(\mu) = r_R(C(\mu))\Lambda \).

2. \( \rightarrow 1. \): Let \( \mu, \nu \in \Lambda \) be such that \( \mu \nu = 0 \). Then by using Lemma 5, \( \nu \in r_\Lambda(\mu) = T\Lambda \) for some right ideal \( T \) of \( R \). Hence, \( \nu(t) \in T \) for each \( t \in \text{supp}(\nu) \). So, \( 0 = \mu c_{\nu(t)} \) and for any \( s \in \text{supp}(\mu) \) we have \( 0 = (\mu c_{\nu(t)})(s) = \alpha_s(\alpha_{-s}(\mu(s))\nu(t)) \). Therefore, \( R \) is a partial skew generalized Armendariz ring.

3. \( \rightarrow 1. \): Let \( \mu, \nu \in \Lambda \) such that \( \mu \nu = 0 \). By assumption, \( l_\Lambda(\nu) = \Lambda B \), for some left ideal \( B \) of \( R \). Then \( \mu \in \Lambda B \). Hence \( \alpha_{-s}(\mu(s)) \in B \subseteq l_\Lambda(\nu) \), for all \( s \in \text{supp}(\mu) \). So, \( \alpha_{-s}(\mu(s))\nu(t) = 0 \) for each \( s \in \text{supp}(\mu) \) and \( t \in \text{supp}(\nu) \).

1. \( \rightarrow 3. \): It is only necessary to show that \( \Psi \) is surjective. Let \( \mu \in \Lambda \). Then we want to show that \( l_\Lambda(\mu) = l_\Lambda(C_\mu) \).

In fact, given \( \nu \in l_\Lambda(\mu) \), we have \( \nu \mu = 0 \). Since \( R \) is a partial skew generalized Armendariz ring, then \( \alpha_{-t}(\nu(t))\mu(s) = 0 \) for each \( s \in \text{supp}(\mu) \) and \( t \in \text{supp}(\nu) \). Hence \( \nu \in l_\Lambda(C_\mu) \).

On the other hand, let \( \nu \in \Lambda \) such that \( \nu C_\mu = 0 \). Thus \( \nu \mu(s) = \alpha_{-t}(\nu(t))\mu(s) = 0 \) for each \( s \in \text{supp}(\mu) \) and \( t \in \text{supp}(\nu) \). So, \( \nu \mu(s) = \sum_{(s,t) \in X_s(\mu,\nu)} \alpha_t(\alpha_{-t}(\nu(t))\mu(s)) = 0 \), and we have that \( \nu \in l_\Lambda(\mu) \). We easily have that for each subset \( T \) of \( \Lambda \),

\[
l_\Lambda(T) = l_\Lambda(\bigcup_{\mu \in T} C_\mu)
\]

We claim that \( l_\Lambda(C_\mu) = \Lambda l_R(C_\mu) \). In fact, let \( \nu \in \Lambda \) be such that
\[ \nu C_{\mu} = 0. \] Then We have that \[ 0 = \nu \mu(s) = \alpha_{-t}(\nu(t))\mu(s) \] for each \( s \in \text{supp}(\mu) \). Thus \( \alpha_{-t}(\nu(t)) \in l_R(C_{\mu}) \), and it follows that \( \nu \in \Lambda l_R(C_{\mu}) \). The other inclusion is trivial. So, \( l_A(T) = \bigcap_{\mu \in T} l_A(C_{\mu}) = \Lambda \bigcap_{\mu \in T} l_R(C_{\mu}) = \Lambda l_R(C_T) \). Therefore \( \Psi \) is surjective. \( \blacksquare \)

Now we are able to prove our main result of this paper.

**Theorem 7** Suppose that \( R, G, \alpha \) and \( \Lambda \) are as in Lemma 5. If \( R \) is a partial skew Armendariz ring then \( R \) is a right (left) zip ring if and only if \( \Lambda \) is a right (left) zip ring.

**Proof.** Suppose that \( \Lambda \) is a right zip ring and \( X \subseteq R \) such that \( r_R(X) = 0 \). Let \( Y = \{c_x \in \Lambda : x \in X \} \) and \( r_A(Y) \neq 0 \). Then there exists \( 0 \neq \mu \in r_A(Y) = \bigcap_{x \in X} r_A(c_x) \). Hence for each \( s \in \text{supp}(\mu) \) and \( x \in X \), we have
\[
0 = (c_x \mu)(s) = \alpha_e(\alpha^{-e}(x) \mu(s)) = x \mu(s) \text{. So, } \mu(s) \in r_R(X) = 0 \text{ for each } \mu \in \text{supp}(\mu) \text{. Hence } \text{supp}(\mu) = \Phi \text{ and it follows that } \mu = 0 \text{ which is a contradiction. Therefore, } r_A(Y) = 0 \text{. Since } \Lambda \text{ is a right zip ring, then there exist a finite subset } Y_0 \subseteq Y \text{, where } Y_0 = \{c_{x_i} : i = 1, 2, \ldots, n \} \text{ such that } r_A(Y_0) = 0 \text{. Using the same procedure above it can be easily shown that } r_R(X_0) = 0 \text{ where } X_0 = \{x_i : i = 1, 2, \ldots, n \} \text{ and } R \text{ is a right zip ring.}

Conversely, Suppose that \( R \) is a right zip ring and \( Y \subseteq \Lambda \) such that \( r_A(Y) = 0 \). Let \( T = C(Y) \) be the content of \( Y \), i.e. \( T = \bigcup_{\mu \in Y} \{\mu(s) : s \in \text{supp}(\mu)\} \). Using Lemma 6 \( r_A(Y) = r_R(T) \Lambda = 0 \), and it follows that \( r_R(T) = 0 \). Since \( R \) is a right zip ring, there exists a finite subset \( T_0 \subseteq T \) such that \( r_R(T_0) = 0 \). So, let \( T_0 = \{t_i : i = 1, 2, \ldots, n \} \) and let \( \mu_{t_i} \in Y \) be such that \( t_i \in \{\mu_{t_i}(s) : s \in \text{supp}(\mu)\} \). Let \( Y_0 \) be the minimal subset of \( Y \) which contains \( \{\mu_{t_i} \in Y : t_i \in T_0 \} \) and it is clear that \( Y_0 \) is a finite subset. Let \( T_1 = \bigcup_{\mu_{t_i}} \{\mu_{t_i}(s) : s \in \text{supp}(\mu_{t_i})\} \). Therefore, \( T_0 \subseteq T_1 \) and \( r_R(T_1) \subseteq r_R(T_0) = 0 \). Using Lemma 6 \( r_A(Y_0) = r_R(T_1) \Lambda = 0 \). Consequently, \( \Lambda \) is a right zip ring. \( \blacksquare \)

**References**

